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WAVEGUIDE WITH ARBITRARY CROSS-SECTION CONSIDERED BY CONFORMAL MAPPING

by F. J. Tischer and H. Y. Yee

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Summary - The arbitrarily-shaped waveguide is transformed by conformal mapping into a rectangular guide filled with a hypothetical medium. The permittivity and the permeability are anisotropic with the longitudinal component dependent on the transverse position. The field distributions and the propagation characteristics are computed for the transformed guide for both TE and TM modes. The field distributions in the original waveguide can be obtained by inverse transformation.

An iterative approximation method is used for deriving a solution of the nonseparable wave equation. As an example, the properties of a coaxial guide are considered by this method and the results compared with known data. AUTHOR

Introduction

Simply shaped electromagnetic waveguides (for example: rectangular, circular, elliptical, and parabolic waveguides) have been investigated extensively. In contrast to that knowledge about waveguides of a general cross-section is rather limited. The analytic solution of the propagation problem in a simply shaped waveguide filled with homogeneous isotropic dielectric can be obtained without difficulties because of the separability of the wave equation for the coordinate system pertinent to the boundary conditions.

Work on waveguides of a more complicated cross-section was started by Cohn(1) in a study of the ridge guide in 1947. Iashkin(2,3) and Yashkin(4) made some first approximate calculations for the cut off frequencies of waveguides with complicated cross-section. Recently, Hu and Ishimaru(5,6) investigated the lunar line in the same way. Clement and Johnson(7) considered the cut off frequencies for waveguides of arbitrary cross section experimentally. However, so far, no analytic solution has been made for waveguides whose surfaces upon which boundary conditions are to be satisfied is not one of the separable coordinate surfaces.*

One way to solve the problem of wave propagation in an arbitrarily shaped waveguide applying conformal mapping is shown by Tischer(8). Some transformations by conformal mapping with equal scale factors of two-dimensional coordinate systems (for example: polar, elliptical, and parabolic coordinate systems) are well known(9). If the cross section of a waveguide under investigation follows the coordinates of one of these systems, it is possible to analyze the properties of the waveguide in the transformed plane (the Z-plane) in which the cross-section has a rectangular shape. Herewith the computation of the properties of an arbitrarily-shaped waveguide filled with uniform and isotropic medium is substituted by that of a rectangular guide filled with nonuniform and anisotropic medium as it will be shown later. This method is then applicable to waveguides of any shape for which the conformal transformation into rectangular coordinate can be found.

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* A very recent article by H. H. Meinke and co-authors in the Proceeding of the IEEE (vol. 51, November 1963, 1436-1443) deals with waveguides of a general cross-section. It was read by the authors after completion of this paper. A different approach of the solution of the wave equation is followed in this paper.

In the following considerations, basic relations between the field components, the permittivity, and the permeability in the Z-space (the Z-space is the space of the Z-plane together with the longitudinal coordinate) and in the W-space (the original space) are derived. First, the case is treated where the wave equation obtained from Maxwell's equations is separable in both spaces (the W-space and the Z-space). The analysis of the coaxial waveguide in the Z-space is made to illustrate that the properties of a waveguide under consideration in the Z-space are the same as those in the W-space. The second case deals with cross-sections which lead to nonseparable wave equations. Perturbation formulas are employed for the computation of the properties propagation of the guide. The approximate cut-off frequencies, guided wave lengths, and field components may be obtained if the square of the scale factor is integrable within the region under consideration. Some of the cut-off frequencies of a coaxial waveguide are calculated by the perturbation method and compared with those computed by the direct method in the W-space. The comparison shows that the perturbation method can be used to find good approximate solutions for waveguides of arbitrary cross-section.

Coordinate Transformations

The application of conformal mapping to the solution of two-dimensional static electromagnetic field problems[10] is well known. Now, conformal mapping is applied to guided-wave problems. Disregarding the longitudinal coordinate, the complicated cross-section of an arbitrary waveguide may be transformed into a rectangle or a pair of parallel lines in the transformed space. Theguide with this cross-section is filled with nonuniform anisotropic medium. Both, the permittivity and the permeability have the same tensor form, and they are functions of the transverse coordinates in the W-space. Fortunately, the tensor has only three elements and only one, namely, the longitudinal element is a function of the transverse position.

Suppose there exists a transformation

$$p = f(u,v)$$
$$q = f(u,v)$$

such that the lines of the cross-section of an arbitrarily shaped hollow-pipe waveguide in the W-plane are loci of constant u,v orthogonal curvilinear coordinates [see Fig. 1(a)] where p and q are the two-dimensional Cartesian coordinates. The connection between the two coordinate systems is usually made by means of the length of an infinitesimal line element

$$ds^2 = dp^2 + dq^2 = (h_1\, du)^2 + (h_2 dq)^2 \,. \tag{1}$$

The rectangular coordinates x and y in the Z-plane are, by definition, functions of only the orthogonal coordinates u and v in the W-plane respectively. Then, the arbitrary shape in the W-plane as shown in Fig. 1(a) may be transformed into a rectangle in the Z-plane as shown in Fig. 1(b).

Comparison of Maxwell's equations for orthogonal curvilinear coordinate system in the W-space and for the rectangular coordinate system in the Z-space shows[8] that for $h_1 = h_2$

$$H^W = \begin{bmatrix} H_u \\ H_v \\ H_z \end{bmatrix}, \qquad H^Z = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix},$$

$$E^W = \begin{bmatrix} E_u \\ E_v \\ E_z \end{bmatrix}, \qquad E^Z = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix},$$

$$\begin{bmatrix} hH_u \\ hH_v \\ H_z \end{bmatrix} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix},$$

$$\begin{bmatrix} hE_u \\ hE_v \\ E_z \end{bmatrix} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \tag{2}$$

$$\mu^Z = \mu^W \begin{bmatrix} h^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\epsilon^Z = \epsilon^W \begin{bmatrix} h^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{2}$$

where the superscripts indicate that the quantity is in the W- or in the Z-space. The other quantities:

$h = h_1 = h_2 =$ the scale factor for the coordinates u and v,

$\mu =$ permeability,

$\epsilon =$ permittivity.

If the original waveguide is bounded by a perfect conductor and filled with air, the wave equation obtained for the rectangular guide in the Z-space is a partial differential equation of the following general form (ψ is a representative scalar):

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 h^2 \psi = 0 , \tag{3}$$

for time varying fields ($e^{j\omega t}$), where

$$k^2 = k_o^2 - k_z^2$$

$$k_o^2 = \omega^2 \mu_o \epsilon_o ,$$

$$k_z^2 = \left(\frac{2\pi}{\lambda_g}\right)^2 .$$

The quantity k_z is the longitudinal propagation constant and λ_g the guide wavelength. The scale factor h is a function of x and y. For TE (transverse electric) wave modes, $\psi = H_z$, for TM (transverse magnetic) wave modes $\psi = E_z$. Whenever a suitable solution $\psi = E_z(x,y)$, $H_z(x,y)$ is found for the boundary conditions [as shown in Fig. 1(b)], the transverse components of the fields are given by

$$E_x = -\frac{jk_z}{k^2}\left(\frac{\partial E_z}{\partial x} + \frac{\omega\mu_o}{k_z}\frac{\partial H_z}{\partial y}\right) , \tag{4}$$

$$E_y = -\frac{jk_z}{k^2}\left(\frac{\partial E_z}{\partial y} - \frac{\omega\mu_o}{k_z}\frac{\partial H_z}{\partial x}\right),$$

$$H_x = \frac{jk_z}{k^2}\left(\frac{\omega\epsilon_o}{k_z}\frac{\partial E_z}{\partial y} - \frac{\partial H_z}{\partial x}\right), \tag{4}$$

$$H_y = -\frac{jk_z}{k^2}\left(\frac{\omega\epsilon_o}{k_z}\frac{\partial E_z}{\partial x} + \frac{\partial H_z}{\partial y}\right),$$

where the factor $e^{j(\omega t - k_z z)}$ for every component is understood. All the components of the fields of the curvilinear coordinates in the W-space may be calculated by Eq. (2).

Transformation and Conformal Mapping

As a next step, a complex relationship between the coordinates in the two coordinate systems is introduced. The cross-sectional coordinates for the assumed waveguides may be represented by a complex function

$$R = f(Z),$$

where

$$R = p + jq,$$

$$Z = x + jq.$$

The cross-sections of the two waveguides are shown in Fig. 1(a) and (b). The orthogonal curvilinear coordinates u and v are contained in the rectangular system in the W-plane, and are functions of only x and y respectively. For regular and analytic functions, it can be shown that under these conditions, the scale factor is given by

$$h = \left|f'(Z)\right| = \left|\frac{dR}{dZ}\right|. \tag{5}$$

The wave equation Eq. (3) is, in the general case, nonseparable. In some special cases, it is separable. The separation is possible if the coefficient h^2, which is a function of x and y, can be written as follows:

$$h^2 = f'(x + jy)^2 = g_1(x) + g_2(y), \tag{6}$$

where g_1 and g_2 are functions of only x and y respectively. Either g_1 and g_2 may vanish. Noting that

$$\left| f'(Z) \right|^2 = f'(Z)\, f'(Z^*) ,$$

where the star stands for complex conjugate quantity. Then, Eq. (6) reduces to

$$\frac{\partial^2}{\partial x \partial y} \left[f'(Z)\, f'(Z^*) \right] = 0 .$$

Introducing

$$\frac{\partial^2}{\partial x \partial y} = i \frac{\partial^2}{\partial Z^2} - i \frac{\partial^2}{\partial Z^{*2}} ,$$

the method of separation yields

$$\frac{d^2}{dZ^2} \left[f'(Z) \right] = \gamma f'(Z) ,$$

$$\frac{d^2}{dZ^{*2}} \left[f'(Z^*) \right] = \gamma f'(Z^*) . \qquad (7)$$

Four two-dimensional coordinate systems (namely: rectangular, polar, parabolic, elliptical-hyperbolic) may be obtained from the solutions of Eq. (7) as tabulated in Table 1.

TABLE 1 - Coordinates as Solutions of Eq. (7).

γ	0		1	
f(Z)	$aZ^2 + bZ + C$		$Ae^{Z} + Be^{-Z}$	
Coordinate	a=0, $b \neq 0$	$a \neq 0$	A=0 or B=0	$A \neq 0$, $B \neq 0$
system	rectangular	Parabolic	Polar	elliptical-hyperbolic

Since the wave equations obtained in the W- and the Z-space are equivalent, the two-dimensional coordinate systems other than the foregoing four are non-separable and the wave equations are non-separable variable partial differential equations.

Mapping of the Cross-Section of a Coaxial Waveguide

As discussed previously, solution of a waveguide may be obtained by considering the waveguide in the Z-space, where the originally complicated cross-section of the waveguide is transformed into a rectangle. Since the branch-points and the periodicity may be introduced by the conformal mapping, care must be taken to consider the boundary conditions such that the corresponding boundary conditions in the Z-space are consistent with those in the W-space. For example, the coaxial waveguide in the W-space may be transformed into a parallel-plane waveguide in the W-space (see Fig. 3) by means of the transformation

$$R = ae^{Z} .$$

It is easily seen that the relations between cylindrical coordinates in the Z-space and the rectangular coordinates in the W-space is given by

$$r = ae^{x} , \quad \alpha = y ,$$

$$p = r \cos \alpha \ , \quad q = r \sin \alpha \ .$$

The shaded area between two concentric circles in the W-plane (for one cycle) corresponds to the shaded rectangle area in the Z-plane as shown in Fig. 2. An additional cycle in the W-plane simply lengthens the rectangle in the Z-plane by 2π in the y-direction. In order to fulfill the boundary conditions and the continuity of the field components of the coaxial waveguide in the W-space, the corresponding rectangular waveguide in the Z-space must satisfy the following boundary conditions:

(A) TE modes:

$$\frac{\partial}{\partial x} \psi (0,y) = \frac{\partial}{\partial x} \psi (b,y) = 0 ,$$

$$\frac{\partial}{\partial y} \psi (x,0) = \frac{\partial}{\partial y} \psi (x,2\pi) , \tag{8}$$

$$\frac{\partial}{\partial x} \psi (x,0) = \frac{\partial}{\partial x} \psi (x,2\pi) ;$$

(B) TM modes:

$$\psi(0,y) = \psi(b,y) = 0,$$

$$\frac{\partial}{\partial y}\psi(x,0) = \frac{\partial}{\partial y}\psi(x,2\pi), \tag{9}$$

$$\frac{\partial}{\partial x}\psi(x,0) = \frac{\partial}{\partial x}\psi(x,2\pi).$$

The scale factor may be calculated by taking the magnitude of the derivative of R with respect to Z as follows:

$$h^2 = a^2 e^{2x}. \tag{10}$$

Substituting (10) into Eq. (3) the wave equation assumes the form

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + K^2 e^{2x}\psi = 0 , \tag{11}$$

where $K^2 = k^2 a^2$.

Since Eq. (11) is a separable-variable partial differential equation, the exact analytic solution can be written as a product function

$$\psi(x,y) = X(x)\,Y(y).$$

Substituting into Eq. (11) and employing the conventional steps yields

$$\frac{d^2 Y}{dy^2} + k_y^2 Y = 0 , \tag{12a}$$

$$\frac{d^2 X}{dx^2} + (K^2 e^{2x} - k_y^2) X = 0 , \tag{12b}$$

where k_y is a separation constant. Solution of Eq. (12a) is readily seen to be

$$Y(y) \sim {\cos \atop \sin} k_y y.$$

By change of variable, Eq. (12b) may be reduced to

$$\frac{d^2X}{d\rho^2} + \frac{1}{\rho}\frac{dX}{d\rho} + \left(\frac{K^2}{4\rho} - \frac{k_y^2}{4\rho^2}\right) X = 0 , \tag{12c}$$

where $\rho = e^{2x}$.

Solution of Eq. (12c) is given by [11]

$$X = AJ_{ky}(Ke^x) + BN_{ky}(Ke^x) .$$

The total solution of Eq. (11) may be expressed in the form:

$$\psi = [AJ_{ky}(Ke^x) + BN_{ky}(Ke^x)] \begin{matrix}\cos\\ \sin\end{matrix} k_y y . \tag{13}$$

The periodicity of the boundary condition along the y-direction requires that $k_y = n$, where n is an integer.

Confirming the problem to the TE mn modes, the field components in the Z-space may be obtained from Eqs. (4) and (13) as follows:

$$\begin{aligned}
H_z &= [AJ_n(Ke^x) + BN_n(Ke^x)] \begin{matrix}\cos\\ \sin\end{matrix}\, ny , \\
H_x &= -\frac{jk_z}{k} ae^x [AJ'_n(Ke^x) + BN'_n(Ke^x)] \begin{matrix}\cos\\ \sin\end{matrix}\, ny , \\
H_y &= \pm \frac{jk_z}{k^2} n [AJ_n(Ke^x) + BN_n(Ke^x)] \begin{matrix}\sin\\ \cos\end{matrix}\, ny , \\
E_z &= 0 , \\
E_x &= \frac{\omega\mu_o}{k_z} H_y , \\
E_y &= -\frac{\omega\mu_o}{k_z} H_x .
\end{aligned} \tag{14a}$$

The value of k may be calculated from the characteristic equation; subject to the boundary conditions (8), it is given by

$$\frac{J'_n(Ke^b)}{J'_n(K)} = \frac{N'_n(Ke^b)}{N'_n(K)} \quad . \tag{14b}$$

By conventional method, the solution of the wave equation in circular cylindrical coordinates leads to[12]

$$H_z = AJ_n(kr) + BN_n(kr) \begin{matrix}\cos\\ \sin\end{matrix}\; n\alpha \quad ,$$

$$H_r = -\frac{jkg}{k} \quad AJ'_n(kr) + BN'_n(kr) \begin{matrix}\cos\\ \sin\end{matrix}\; n\alpha \quad ,$$

$$H_\alpha = \pm \frac{jkg}{k^2}\frac{n}{r} \quad AJ_n(kr) + BN_n(kr) \begin{matrix}\sin\\ \cos\end{matrix}\; n\alpha \quad , \tag{15a}$$

$$E_z = 0 \quad ,$$

$$E_r = \frac{\omega\mu_o}{kg} \quad H_\alpha \quad ,$$

$$E_\alpha = \frac{\omega\mu_o}{kg} \quad H_r \quad ,$$

and

$$\frac{J'_n(kr_2)}{J'_n(kr_1)} = \frac{N'_n(kr_2)}{N'_n(kr_1)} \quad . \tag{15b}$$

Observe that the Eqs. (14) and (15) become equal if the following relations are introduced, i.e., $r_1 = a$, $r_2 = ae^b$, $r = h = ae^x$, $\alpha = y$, $H_r = H_u$, $H_\alpha = H_v$; and Eqs. (14) are substituted into Eqs. (2).

This shows that the conformal-mapping method leads to the same field equations as they are obtained by the direct solution of the wave equation in the cross-sectional coordinate system taking into account the boundary conditions.

Approximate Solution of the Nonseparable Wave Equation

In general, the wave equation is nonseparable and no exact solution for arbitrary boundary conditions is known. Therefore, approximate techniques have to be used to solve this problem. It should be noted that even for the separable wave equation, which can be solved exactly, the approximate method of evaluation may be simpler in some cases. In the following sections, a perturbation method and an example of its application will be presented.

Consider that the scalar function ψ of Eq. (3) is limited to the region $0 \leq x \leq a$, $0 \leq y \leq b$, and subject to the boundary conditions

$$\frac{\partial}{\partial x} \psi(0,y) = \frac{\partial \psi(a,y)}{\partial x} = \frac{\partial \psi(x,0)}{\partial y} = \frac{\partial \psi(x,b)}{\partial y} = 0, \qquad (16a)$$

which is typical for TE modes in waveguides. The corresponding conditions for TM modes are

$$\psi(0,y) = \psi(a,y) = \psi(x,0) = \psi(x,b) = 0. \qquad (16b)$$

Define next a complete set of functions $\{\phi_q\}$, where ϕ_q satisfies the foregoing boundary conditions, as ψ does, and where ϕ_q may have the same form as one of the eigenfunctions of the wave equation with h^2 constant. The function ϕ_q has hence the form:

$$TE_{mn}: \quad \phi_q^{(1)} = \sqrt{\frac{\epsilon_m \epsilon_n}{ab}} \cos \frac{m\pi}{a} x \cos \frac{n\pi}{b} y, \qquad (17a)$$

$$TM_{mn}: \quad \phi_q^{(2)} = \sqrt{\frac{4}{ab}} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y, \qquad (17b)$$

where $\epsilon_{m,n} = 1$, if $m,n = 0$ and $\epsilon_{m,n} = 2$, if $m,n \neq 0$. The subscript q is used to denote the general indices

$$m,n = 0, 1, 2, 3, 4, \ldots\ldots\ldots$$

The expansion of one eigenfunction ψ_p (for variable coefficient in terms of ϕ_q may be written as

$$\psi_p^{(1),(2)} = \sum_q A_q^{(1),(2)} \phi_q^{(1),(2)}, \qquad (18)$$

where p indicates specific pairs of m,n for TE or TM modes. Substituting Eq. (18) into Eq. (3), carrying out the differentiation, rearranging the terms, and forming the integral over the cross-section yields

$$\sum_q \int_0^a \int_0^b \phi_r \left[- L_q^2 + k_p^2 h^2 (x,y) \right] A_q \phi_q \, dxdy = 0 \quad , \tag{19a}$$

where $L_q^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$.

The constant k_p is the eigenvalue for the eigenfunction $_p$. The integration gives

$$\sum_q \left(- L_q^2 \, \delta_{rq} + k_p^2 B_{rq} \right) A_q = 0 \, , \tag{19b}$$

where $\int_0^a \int_0^b \phi_r \phi_q = \delta_{rq}$,

$$B_{rq} = \int_0^a \int_0^b \phi_r h^2 (x,y) \phi_q \, dxdy \, , \tag{20}$$

$$\delta_{rq} = \begin{cases} 1 & \text{if } r = q \\ 0 & \text{if } r \neq q \end{cases} \quad .$$

The nontrivial solution of the scalar function can be obtained only if the determinant formed from the terms between the parentheses of Eq. (19b) vanished. Hence

$$\det \left| L_q^2 \, \delta_{rq} - k_p^2 B_{rq} \right| = 0 \quad . \tag{21}$$

Eq. (21) can be explored for obtaining the eigenvalues k_p. This is not done here, however, since the computation of these eigenvalues and of the coefficients A_q can be combined evaluating Eq. (19b). The development of the corresponding equation is shown in the Appendix. If the outlined procedure is repeated an infinite times, it results in the following equations for A_q and k_p:

$$A_q = \frac{B_{qp}}{D_{pq}} + \sum_{r \neq q,p} \frac{B_{qr} B_{rp}}{D_{pq} D_{pqr}} + \sum_{\substack{r \neq q,p \\ s \neq r,q,p}} \frac{B_{qr} B_{rs} B_{sp}}{D_{pq} D_{pqr} D_{pqrs}} + \dots , \tag{22}$$

$$A_p = 1 ,$$

$$k_p^2 = L_p^2 / \{ B_{pp} + \sum_{q \neq p} \frac{B_{pq} B_{qp}}{D_{pq}} + \sum_{\substack{q \neq p \\ r \neq q,p}} \frac{B_{pq} B_{qr} B_{rp}}{D_{pq} D_{pqr}} + \dots . \} , \tag{23}$$

where $D_{pq} = \frac{L_q^2}{k_p^2} - (B_{qq} + \sum_{r \neq q,p} \frac{B_{qr} B_{rq}}{D_{pqr}} + \sum_{\substack{r \neq q,p \\ s \neq r,q,p}} \frac{B_{qr} B_{rs} B_{sq}}{D_{pqr} D_{pqrs}} + \dots .),$

$$D_{pqr} = \frac{L_r^2}{k_p^2} - (B_{rr} + \sum_{s \neq r,q,p} \frac{B_{rs} B_{sr}}{D_{pqrs}} + \sum_{\substack{s \neq r,q,p \\ t \neq s,r,q,p}} \frac{B_{rs} B_{st} B_{tr}}{D_{pqrs} D_{pqrst}} + \dots .)$$

The notation $\sum_{r \neq q,p}$ means that the terms $r = q$ and $r = p$ are omitted from the sum, etc. For solving the Eq. (23) the same iterative method is used as outlined in the Appendix, the resulting approximation of the various orders are given in the following:

First order: $(k_p^2)^{(1)} = L_p^2 / B_{pp}$,

Second order:

$$(k_p^2)^{(2)} = L_p^2 / \{ B_{pp} + \sum_{q \neq p} \frac{B_{pq} B_{qp}}{\frac{L_q^2}{(k_p^2)^{(1)}} - B_{qq}} \} , \tag{24}$$

Third order:

$$(k_p^2)^{(3)} = L_p^2 \Bigg/ \left\{ B_{pp} + \sum_{q \neq p} \frac{B_{pq} B_{qp}}{\dfrac{L_q^2}{(k_p^2)^{(2)}} - \left(B_{qq} + \displaystyle\sum_{r \neq p,q} \frac{B_{qr} B_{rq}}{\dfrac{L_r^2}{(k_p^2)^{(2)}} - B_{rr}} \right)} \right.$$

$$\left. + \sum_{\substack{q \neq p \\ r \neq q,p}} \frac{B_{pq} B_{qr} B_{rp}}{\left[\dfrac{L_q^2}{(k_p^2)^{(2)}} - \left(B_{qq} + \displaystyle\sum_{r \neq p,q} \frac{B_{qr} B_{rq}}{\dfrac{L_r^2}{(k_p^2)^{(2)}} - B_{rr}} \right) \right] \left[\dfrac{L_r^2}{(k_p^2)^{(2)}} - B_{rr} \right]} \right\},$$

Fourth and higher order values of k_p^2 can be developed correspondingly.

Substituting the approximation of a specified order for k_p into Eq. (22) and disregarding terms of higher order yield the corresponding approximation for A_q. In this manner cutoff-frequencies k_c ($k_c = k_p$) and field distribution in the Z-space can be computed with any accuracy. The components of the fields in the original W-space (arbitrary cross-section) may be obtained by using Eq. (2).

Cutoff Frequencies of a Coaxial Waveguide Determined by Perturbation Equations

As an example, the cutoff frequencies of a coaxial waveguide are computed by using the derived perturbation equations and compared with those obtained by direct calculation.

The coaxial waveguide under consideration is the same guide (see Fig. 2) treated previously for comparing the results obtained by conformal mapping with those resulting from direct calculation. Consider the TE modes first in the Z-space. The function ψ is subject to the boundary conditions (11). Modification of Eq. (17a) gives

$$\phi_{mn} = \sqrt{\frac{\epsilon_m \epsilon_n}{2\pi b}} \cos \frac{m\pi}{b} x \begin{matrix} \cos \\ \sin \end{matrix} ny .$$

Substituting into Eq. (20) and evaluating the integration yield

$$B_{oo} = a^2 \frac{e^{2b} - 1}{2b} ,$$

$$B_{om} = B_{mo} = a^2 \frac{\sqrt{8}}{b} \frac{(-1)^m e^{2b} - 1}{4 + (\frac{m\pi}{b})^2} ,$$

$$B_{mm} = a^2 \frac{e^{2b} - 1}{2b} \frac{2 + (\frac{m\pi}{b})^2}{1 + (\frac{m\pi}{b})^2} ,$$

and

$$B_{mm'} = a^2 \frac{2}{b} \left[(-1)^{m+m'} e^{2b} - 1 \right] \left[\frac{1}{4 + (\frac{m+m'}{b}\pi)^2} + \frac{1}{4 + (\frac{m-m'}{b}\pi)^2} \right] ,$$

where q stands for m,n and r for m',n'. If $n \neq n'$, it follows that $B_{qr} = B_{mnm'n'} = 0$. For the case that $e^b = 3$, or b = 1.0986, the eigenvalues of the first and second order of approximation ka, calculated by Eq. (24) for TE_{10} and TE_{20} modes are tabulated in Table II. They are compared with the exact values.

Table II - Comparison of the Propagation Constants

	$(ka)^{(1)}$	$(ka)^{(2)}$	Exact
TE_{10}	1.4231	1.6344	1.6355
TE_{20}	2.955	3.1744	3.1785

It is noted that the convergence is very good, and the values of ka approach fast the exact value.

Turning to the TM modes, the solution of ψ is subject to the boundary condition (12), so that

$$\phi_{mn} = \sqrt{\frac{2}{\pi b}} \sin \frac{m\pi}{b} x \begin{matrix} \cos \\ \sin \end{matrix} ny ,$$

$$B_{mm} = a^2 \frac{e^{2b} - 1}{2b} \frac{(\frac{m\pi}{b})^2}{1 + (\frac{m\pi}{b})^2} ;$$

$$B_{mm} = a^2 \frac{2}{b} \left[(-1)^{m+m'} e^{2b} - 1 \right] \left[\frac{1}{4 + (\frac{m-m'}{b} \pi)^2} - \frac{1}{4 + (\frac{m+m'}{b} \pi)^2} \right] ;$$

If $n \neq n'$ $B_{mnm'n'} = 0$. For the same waveguide as before, i.e., b = 1.0986, the eigenvalue ka calculated by Eq. (24) for the TM_{11} mode is tabulated in Table III and compared with the exact value.

Table III - Comparison of the Propagation Constants

	$(ka)^{(1)}$	$(ka)^{(2)}$	Exact
TM_{11}	1.6819	1.6339	1.6355

Conclusion

The foregoing consideration of conformal mapping shows that this method is useful for computing the field distributions and propagation characters of waveguide with arbitrary cross-section. Perturbation method can be used for determination of the field distribution in the guide and its characteristics in the transformed plane in which the cross-section is a rectangle. These quantities are re-transformed into the original plane with the original cross-section. The example shows that good approximation is obtained after a few iterative steps of the successive approximation.

Appendix. Development of the Perturbation Equation

Suppose that the waveguide in the W-space changes its cross-section to become a rectangle, the solution ψ_p reduces to ϕ_p, so that $A_p = 1$. Rewriting Eq. (19b) and bringing the term $q = p$ (with $A_p = 1$) to right-hand side yields

$$\sum_{q \neq p} \left(\frac{L_q^2}{k_p^2} \delta_{rq} - B_{rq} \right) A_q = B_{rp} - \frac{L_p^2}{k_p^2} \delta_{rp} . \tag{A.1}$$

The new notation $\sum_{q \neq p}$ means that the term $q = p$ is omitted from the sum. In some practical cases, the off diagonal elements of the matrix

$$\left[\frac{L_q^2}{k_p^2} \delta_{rq} - B_{rq} \right]$$

are very small, i.e., $B_{rq} \ll B_{rr}$ if $r \neq p$*. Evaluation of the equation system Eq. (A.1) shows that $A_q \ll 1$, where $q \neq p$.

As an approximation it is hence assumed that $A_q \approx 0$, $B_{rq} \approx 0$ if $|m - m'| \gg 1$ and $|n - n'| \gg 1$. Then Eq. (19b) reduces to a finite-number equation system with a finite number of unknowns. The perturbation method can be started hence with a finite number of values A_q. Eq. (19b) may be written as (with $A_p = 1$, $r = p$)

$$L_p^2 - k_p^2 B_{pp} - k_p^2 \sum_{q \neq p} B_{pq} A_q = 0. \tag{A.2}$$

To determine A_p, the terms in p, and q are separated out in the sum of Eq. (19b) as

$$(L_q^2 - k_p^2 B_{qq}) A_q - k_p^2 (B_{qp} + \sum_{r \neq q, p} B_{qr} A_r) = 0. \tag{A.3}$$

As a first step, assume that $A_q = 0$. From Eq. (A.2), the first order approximation for the eigenvalue k_p becomes

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* The scale factor h of a rectangular waveguide in the W-space is a constant, namely, h_o. Then $B_{rp} = h_o^2 \delta_{rp}$, $A_q = 0$; hence a waveguide not far away from the rectangular guide will satisfy this assumption. However, in general, $B_{rq} \ll B_{rr}$ if the order numbers differ greatly.

$$k_p^2 = L_p^2 / B_{pp} . \tag{A.4}$$

Next if it is assumed that $A_r = 0$, then, solving Eq. (A.3) for A_q yields the first order approximation for the coefficient

$$A_q = \frac{B_{qp}}{\dfrac{L_q^2}{k_p^2} - B_{qq}} \tag{A.5}$$

Substituting Eq. (A.5) into (A.2), one obtains the second order approximation for k^2:

$$k_p^2 = L_p^2 / \; B_{pp} + \sum_{q \neq p} \frac{B_{pq} B_{qp}}{\dfrac{L_q^2}{k_p^2} - B_{qq}} \quad . \tag{A.6}$$

Continuation of this procedure toward higher order terms gives for the second order value of A_q

$$A_q = \frac{B_{qp}}{\dfrac{L_q^2}{k_p^2} - B_{qq} - \sum\limits_{r \neq q,p} \dfrac{B_{qr} B_{rq}}{\dfrac{L_r^2}{k_p^2} - B_{rr}}} + \sum_{r \neq q,p} \frac{B_{qr} B_{rp}}{\left[\dfrac{L_q^2}{k_p^2} - B_{qq} - \sum\limits_{r \neq qp} \dfrac{B_{qr} B_{rq}}{\dfrac{L_r^2}{k_p^2} - B_{rr}}\right]\left[\dfrac{L_r^2}{k_p^2} - B_{rr}\right]} \, .$$

Further repetition leads to the general solution given by Eqs. (22) and (23).

In the case that the solutions of the wave equation are degenerate, e.g., when the eigenfunctions ψ_p, ψ_{p_1}, $\psi_{p_2} \ldots \psi_{p_n}$ have the same eigenvalue k_p, Eq. (A.1) is not valid to solve for the coefficient A_p, and the series of Eqs. (22) and (23) are not convergent. This difficulty may be overcome by modification of the expansion of the eigenfunction.

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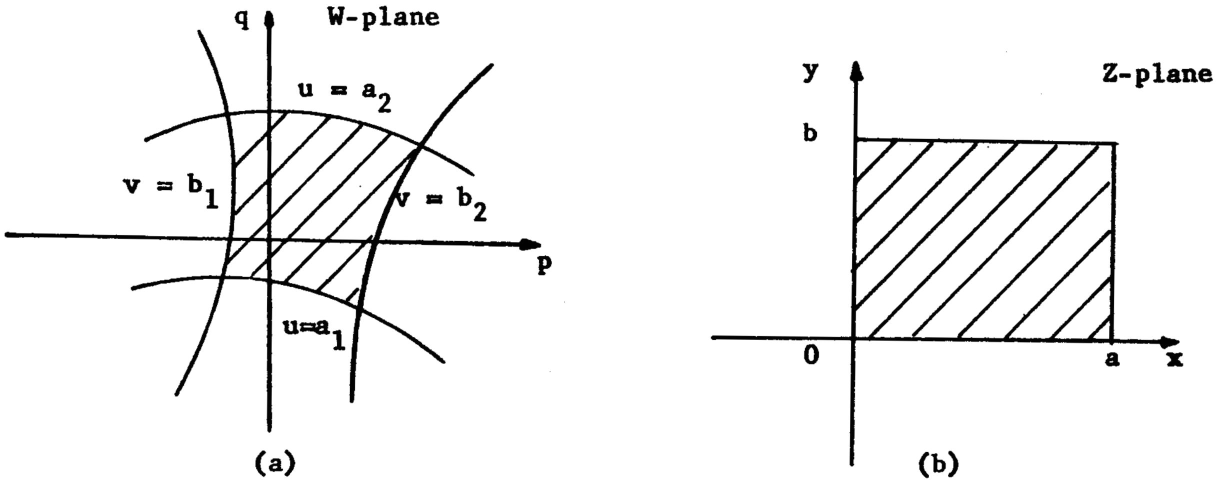

Fig. 1 - Cross-section of a waveguide in the corresponding plane. (a) Arbitrary cross-section in the W-plane. (b) The corresponding cross-section in the Z-plane.

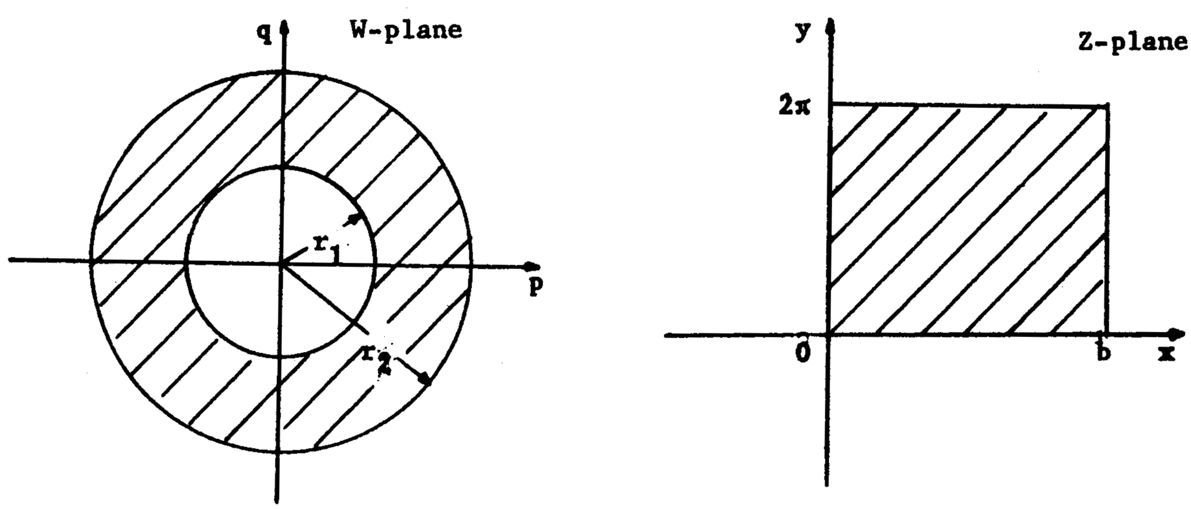

Fig. 2 - The corresponding cross-sections of a coaxial waveguide in the W- and Z-plane.